Preprint IPM-96

A regularizing commutant duality for a kinematically covariant partial ordered net of observables [1]

Martin Rainer[2] §† and **Hadi Salehi** [3] §‡

§Institute for Studies in Physics and Mathematics
P.O.Box 19395-5531, Tehran, Iran

†Institut für Mathematik
Universität Potsdam, PF 601553
D-14415 Potsdam, Germany

‡Arnold Sommerfeld Institute for Mathematical Physics
TU Clausthal, Leibnizstr. 10
D-38678 Clausthal-Zellerfeld, Germany

Abstract

We consider a net of $*$-algebras, locally around any point of observation, equipped with a natural partial order related to the isotony property. Assuming the underlying manifold of the net to be a differentiable, this net shall be kinematically covariant under general diffeomorphisms. However, the dynamical relations, induced by the physical state defining the related net of (von Neumann) observables, are in general not covariant under all diffeomorphisms, but only under the subgroup of dynamical symmetries.

We introduce algebraically both, IR and UV cutoffs, and assume that these are related by a commutant duality. The latter, having strong implications on the net, allows us to identify a 1-parameter group of the dynamical symmetries with the group of outer modular automorphisms.

For thermal equilibrium states, the modular dilation parameter may be used locally to define the notions of both, time and a causal structure.

[1] *This work was financially supported by a DAAD fellowship (M.R.).*

[2] e-mail: mrainer@aip.de, mrainer@physics.ipm.ac.ir

[3] e-mail: salehi@netware2.ipm.ac.ir

About 30 years ago Ekstein [1] introduced the concept of *presymmetry*, as the remaining kinematical effect of covariance (there called space-time symmetry) for the *observation proceedures* even when it is broken for the *observables*.

The observation proceedures represent the abstract kinematical framework for possible preparations of measurements, while the observables encode the kinds of questions we can ask from the physical system. It is intuitively clear, that the same question can be asked in many different forms, i.e. in general there are many possible preparations of measurement for the same observable.

The covariance group of the observation proceedures reflects their general structure. The more sophisticated the structure of the observation proceedures, the smaller the covariance group will be in general. E.g. in [2] the kinematical observation proceedures are given by a network of discrete vertices of a specific Riemannian surface embedded in a $3+1$-dimensional space-time M, whence the covariance group is only that subgroup of $\mathrm{Diff}(M)$ which leaves this structure invariant. In general it is a difficult question, how much structure might be put on the observation proceedures.

In a concrete observation the kinematical covariance will be broken. So in [2] a concrete local observation requires the explicit selection of one of many apriori equivalent vertices, whence it breaks the covariance which holds for the network of vertices as a whole. In the examples of [1] the kinematical covariance was assumed to be broken in a concrete observation by a dynamical interaction with external fields.

We may say that a presymmetry exists if, irrespectively of the loss of covariance in a concrete observation, the action of the covariance group is still welldefined on the observation proceedures. In any case, the loss of covariance in a concrete observation is related to a specific structure of the state of the physical system. Hence, in the following we consider the loss of general covariance as directly induced by the physical state itself.

Let us examine now the consequences of this breaking of general covariance within an algebraic approach to generally covariant quantum field theory, which has been proposed in [3] and further considered in [4, 5]. From the principle of locality, which is at the heart of the standard algebraic approach to quantum field theory [6], we keep the assertions that the observation procedures correspond to possible preparations of localized measurements in finite regions. However, here we do not want to specify a notion of time or a causal structure a priori. It was shown in [7] for a net of subalgebras of a Weyl algebra that, it is indeed possible to work with a flexible notion of causality rather than a rigidly given one. In the same spirit, here we do not impose any a priori causal relations between observables on different regions.

In principle, it is even possible [10] to construct a net of algebras, together with its underlying Hausdorff topological space M, by the partial order via inclusion of (the set of subsets of) the algebras themselves. Although we find this approach, where the net and its underlying manifold M are derived just from the algebras, very appealing, it is beyond the scope of the present investigations. For our examination on the dynamical symmetries we need a differentiable structure on M. It might be, that even this structure can be derived in not too ambigious manner (cf. some recent discussion in [**?**]) with the help of some algebraic methods related to noncommutative geometry [15], help one. However, in the present approach we just work a priori with a net of $*$-algebras on an

underlying differentiable manifold M.

Such a net net associatiates to each open set $\mathcal{O} \in M$ a $*$-algebra $\mathcal{A}(\mathcal{O})$ such that isotony,

$$\mathcal{O}_1 \subset \mathcal{O}_2 \Rightarrow \mathcal{A}(\mathcal{O}_1) \subset \mathcal{A}(\mathcal{O}_2), \tag{1}$$

holds ($\subset$ here always denotes a proper, nonidentical inclusion). Selfadjoint elements of $\mathcal{A}(\mathcal{O})$ may be interpreted as *observation procedures*, i.e. possible prescriptions for laboratory measurements in $\mathcal{O}$.

There should not be any a priori relations between observation procedures associated with disjoint regions. In other words, the net $\mathcal{A} := \bigcup_{\mathcal{O}} \mathcal{A}(\mathcal{O})$ has to be free from any relations which exceed its mere definition.

This interpretation allows us to extend the Diff(M) covariance from the underlying manifold M to the net of algebras, on which Diff(M) then acts by automorphisms, i.e. each diffeomorphism $\chi \in$ Diff(M) induces an automorphism α_χ of the observation proceedures such that

$$\alpha_\chi(\mathcal{A}(\mathcal{O})) = \mathcal{A}(\chi(\mathcal{O})). \tag{2}$$

The state of a physical system is mathematical described by a positive linear functional ω on $\mathcal{A}$. Given the state ω, one gets via the GNS construction a representation π^ω of $\mathcal{A}$ by a net of operator algebras in a Hilbert space $\mathcal{H}^\omega$ with a cyclic vector $\Omega^\omega \in \mathcal{H}^\omega$. The GNS representation $(\pi^\omega, \mathcal{H}^\omega, \Omega^\omega)$ of any state ω has a socalled folium $\mathcal{F}^\omega$, given as the family of those states $\omega_\rho := \mathrm{tr}\rho\pi^\omega$ which are defined by positive trace class operators ρ on $\mathcal{H}^\omega$.

Once a physical state ω has been specified, one can consider in each algebra $\mathcal{A}(\mathcal{O})$ the equivalence relation

$$A \sim B \quad :\Leftrightarrow \;\; \omega'(A - B) = 0, \quad \forall \omega' \in \mathcal{F}^\omega. \tag{3}$$

These equivalence relations generate a two-sided ideal $\mathcal{I}^\omega(\mathcal{O}) := \{a \in \mathcal{A}(\mathcal{O}) | \omega'(A) = 0\}$ in $\mathcal{A}(\mathcal{O})$. The algebra of observables $\mathcal{A}^\omega_{\mathrm{obs}}(\mathcal{O}) := \pi^\omega(\mathcal{A}(\mathcal{O}))$ may be constructed from the algebra of observation procedures $\mathcal{A}(\mathcal{O})$ by taking the quotient

$$\mathcal{A}^\omega_{\mathrm{obs}}(\mathcal{O}) := \mathcal{A}(\mathcal{O})/\mathcal{I}^\omega(\mathcal{O}). \tag{4}$$

Since any diffeomorphism $\chi \in$ Diff(M) induces an automorphism α_χ of the observation proceedures, one may ask whether, for a given state ω, the action of α_χ will leave the net $\mathcal{A}^\omega_{\mathrm{obs}} := \bigcup_{\mathcal{O}} \mathcal{A}^\omega_{\mathrm{obs}}(\mathcal{O})$ of observables invariant, with an action of the form

$$\alpha_\chi(\mathcal{R}^\omega_{\mathrm{obs}}(\mathcal{O})) = \mathcal{R}^\omega_{\mathrm{obs}}(\chi(\mathcal{O})). \tag{5}$$

In order for this to be possible, the ideal $\mathcal{I}^\omega(\mathcal{O})$ must be transformed covariantly, i.e. the diffeomorphism χ must satisfy

$$\alpha_\chi(\mathcal{I}^\omega(\mathcal{O})) = \mathcal{I}^\omega(\chi(\mathcal{O})). \tag{6}$$

Hence, the algebra of observables, constructed with respect to the folium $\mathcal{F}^\omega$, does no longer exhibit the kinematical Diff(M) symmetry of the observation proceedures. The

symmetry of the observables is dependent on (folium of) the state ω. Therefore, the selection of a folium of states $\mathcal{F}^\omega$, induced by the actual choice of a state ω, results immediately in a breaking of the Diff(M) symmetry. The resulting effective symmetry group, also briefly called the *dynamical group* of the state ω, is given by the subgroup of those diffeomorphisms which satisfy the constraint condition (6). An automorphisms α_χ is called *dynamical* (w.r.t. the given state ω) if it satisfies (6).

The remaining dynamical symmetry group, depending on the folium $\mathcal{F}^\omega$ of states related to ω, has two main aspects which we have to examine if we actually want to specify the physically admissible states: Firstly, it is necessary to specify its state dependent automorphic algebraic action on the net of observables. Secondly, we have to find a geometric interpretation for the group and its action on M.

If we consider the dynamical group as an *inertial*, and therefore global, manifestation of dynamically ascertainable properties of observables, then its (local) action should be correlated with (global) operations on the whole net of observables. This implies that at least some of the dynamical automorphisms α_χ are not inner. (For the case of causal nets of algebras it was actually already shown in [8] that, under some additional assumptions, the automorphisms of the algebras are in general not inner.)

Note that we might consider instead of the net of observables $\mathcal{A}^\omega_{\text{obs}}$ the net of associated von Neumann algebras $\mathcal{R}^\omega_{\text{obs}}(\mathcal{O})$, which can be defined even for unbounded $\mathcal{A}^\omega_{\text{obs}}$, if we take from the modulus of the von Neumann closure $\mathcal{A}^\omega_{\text{obs}}{}''$ all its spectral projections [3]. Then the isotony (1) induces a likewise isotony of on the net $\mathcal{R}^\omega_{\text{obs}} := \bigcup_{\mathcal{O}} \mathcal{R}^\omega_{\text{obs}}(\mathcal{O})$ of von Neumann observables.

In the following we want to exhibit a possibility to introduce in an algebraic manner both IR and UV cutoff regularizations simultaneously, together with a partial ordering on the net of von Neumann observables. Let us consider nonzero open sets $\mathcal{O}^x_s$, located around an arbitrary point $x \in M$, and continuously parametrized by a real parameter s with $0 < s < \infty$ such that

$$s_1 < s_2 \Rightarrow \mathcal{O}^x_{s_1} \subset \mathcal{O}^x_{s_2} \tag{7}$$

and

$$s \to 0 \Rightarrow \mathcal{O}^x_s \to \emptyset. \tag{8}$$

On open sets with parameter s restricted to $0 < s_{\min} < s < s_{\max} < \infty$, the isotony property implies that

$$\mathcal{R}^\omega_{\text{obs}}(\mathcal{O}^x_{s_{\min}}) \subset \mathcal{R}^\omega_{\text{obs}}(\mathcal{O}^x_s) \subset \mathcal{R}^\omega_{\text{obs}}(\mathcal{O}^x_{s_{\max}}). \tag{9}$$

The key step is now to impose a commutant duality relation between the inductive limits given by the minimal and maximal algebras,

$$\mathcal{R}^\omega_{\text{obs}}(\mathcal{O}^x_{s_{\min}}) = \mathcal{R}^\omega_{\text{obs}}(\mathcal{O}^x_{s_{\max}})', \tag{10}$$

where $\mathcal{R}'$ denotes the commutant of $\mathcal{R}$. Then the bicommutant theorem ($\mathcal{R}'' = \mathcal{R}$) implies that likewise also

$$\mathcal{R}^\omega_{\text{obs}}(\mathcal{O}^x_{s_{\max}}) = \mathcal{R}^\omega_{\text{obs}}(\mathcal{O}^x_{s_{\min}})'. \tag{11}$$

If we now demand that all maximal (or all minimal) algebras are isomorphic to each other, independently of the choice of x and the open set $\mathcal{O}^x_{s_{\max}}$ (resp. $\mathcal{O}^x_{s_{\min}}$), then by (10) (resp. (11)) also all minimal (resp. maximal) algebras are isomorphic to each other. We then denote the universal minimal resp. maximal algebra as $\mathcal{R}^\omega_{\min}$ and $\mathcal{R}^\omega_{\max}$ respectively. Note that the duality (10) implies that $\mathcal{R}^\omega_{\min}$ is Abelian, and $\mathcal{R}^\omega_{\max}$ has necessarily a nontrivial center within $\mathcal{R}^\omega_{\text{obs}}$.

By isotony and (7), the mere existence of $\mathcal{R}^\omega_{\min}$ resp. $\mathcal{R}^\omega_{\max}$ fixes already a common size (as measured by the parameter s) of all sets $\mathcal{O}^x_{s_{\min}}$ resp. $\mathcal{O}^x_{s_{\max}}$ independently of $x \in M$. So in this case $s_{\min}$ and $s_{\max}$ really denote an universal short resp.large scale cutoff.

The number $s \in]s_{\min}, s_{\max}[$ parametrizes the partial order of the net of algebras spanned between the inductive limits $\mathcal{R}^\omega_{\min}$ and $\mathcal{R}^\omega_{\max}$. In our theory, where the lower end of the net is Abelian, observations on minimal regions are expected to be rather classical, while, for increasing size, quantum (field) theory might be rather nontrivial. In [8] it was shown that for causal nets the algebras of QFT are not Abelian and not finite-dimensional. The Abelian character of algebras at the lower end might find a natural explanation in a classical, rather than a full QFT behaviour, at short distances. For gravity it has been indeed proposed that, at (ultra-)short distances, it might be described in terms of an underlying classical kinetical theory.

If we consider the algebraic UV and IR cutoffs as introduced above, it should be clear that only those regions (20) of size $s \in [s_{\min}, s_{\max}]$ are admissable for measurement. The commutant duality between $\mathcal{R}^\omega_{\min}$ and $\mathcal{R}^\omega_{\max}$ inevitably yields large scale correlations in the structure of any physical state ω on any admissable region $\mathcal{O}^x_s$ of measurement at x. Indeed, by isotony, the annihilation of the GNS vector Ω^ω in $\mathcal{R}^\omega_{\max}$ implies automatically its likewise annihilation on $\mathcal{R}^\omega_{\min} = {\mathcal{R}^\omega_{\max}}'$. Hence, if Ω^ω is cylic for $\mathcal{R}^\omega_{\max}$, and hence separating for ${\mathcal{R}^\omega_{\max}}'$, it should also be separating for $\mathcal{R}^\omega_{\max}$. So Ω^ω is a cyclic and separating vector for $\mathcal{R}^\omega_{max}$, and by isotony also for any local von Neumann algebra $\mathcal{R}^\omega_{\text{obs}}(\mathcal{O}^x_s)$.

As a further consequence, on any region $\mathcal{O}^x_s$, the Tomita operator S and and its conjugate F can be defined densely by

$$S A \Omega^\omega := A^* \Omega^\omega \quad \text{for} \quad A \in \mathcal{R}^\omega_{\text{obs}}(\mathcal{O}^x_s) \tag{12}$$

$$F B \Omega^\omega := B^* \Omega^\omega \quad \text{for} \quad B \in \mathcal{R}^\omega_{\text{obs}}(\mathcal{O}^x_s)'. \tag{13}$$

The closed Tomita operator S has a polar decomposition

$$S = J \Delta^{1/2}, \tag{14}$$

where J is antiunitary and $\Delta := FS$ is the self-adjoint, positive modular operator. The Tomita-Takesaki theorem [14] provides us with a one-parameter group of state dependent automorphisms α^ω_t on $\mathcal{R}^\omega_{\text{obs}}(\mathcal{O}^x_s)$, defined by

$$\alpha^\omega_t(A) = \Delta^{-it}\ A\ \Delta^{it}, \quad \text{for} \quad A \in \mathcal{R}^\omega_{\max}. \tag{15}$$

So, as a consequence of commutant duality and isotony assumed above, we obtain here a strongly continous unitary implementation of the modular group of ω, which is defined by the 1-parameter family of automorphisms (15), given as conjugate action of operators

$e^{-it\ln\Delta}$, $t \in \mathbb{R}$. By (15) the modular group, for a state ω on the net of von Neumann algebras, defined by $\mathcal{R}^{\omega}_{\max}$, might be considered it as a 1-parameter subgroup of the dynamical group. Note that, with Eq. (13), in general, the modular operator Δ is not located on $\mathcal{O}^x_s$. Therefore, in general, the modular automorphisms (15) are not inner. It is known (see e.g. [13]) that the modular automorphisms act as inner automorphisms, iff the von Neumann algebra $\mathcal{R}^{\omega}_{\text{obs}}(\mathcal{O}^x_s)$ generated by ω contains only semifinite factors, i.e. factors of type I and II. In this case ω is a semifinite trace.

Above we considered concrete von Neumann algebras $\mathcal{R}^{\omega}_{\text{obs}}(\mathcal{O}^x_s)$, which are in fact operator representations of an abstract von Neumann algebra $\mathcal{R}$ on a GNS Hilbertspace $\mathcal{H}^{\omega}$ w.r.t. a faithful normal state ω. In general, different faithful normal states generate different concrete von Neumann algebras and different modular automorphism groups of the same abstract von Neumann algebra. Let ω_1 and ω_2 be two different faithful normal states on a von Neumann algebra $\mathcal{R}$, and Ω_1 resp. Ω_2 the corresponding cyclic and separating vectors in the corresponding GNS representation π_1 resp. π_2. Then the unitary cocycle theorem [13] asserts that there exists a strongly continuous 1-parameter family of unitary operators $U(t)$, which satisfy the cocycle condition

$$U(t+s) = U(t)\alpha_t^{\omega_1}(U(s)) \quad \text{for} \quad t, s \in \mathbb{R}, \tag{16}$$

and relate the modular group of ω_2 to that of ω_1,

$$\alpha_t^{\omega_2}(A) = U(t)\alpha_t^{\omega_1}(A)U^*(t) \quad \text{for} \quad A \in \mathcal{R},\ t \in \mathbb{R}. \tag{17}$$

Any two modular groups related by (17) are called outer equivalent. The unitarities $U(t)$ are are in fact given as $U(t) := e^{-itH_{\Omega_1,\Omega_2}}$ with a relative Hamiltonian

$$H_{\Omega_1,\Omega_2} := \ln(\Delta_{\Omega_1,\cdot}/\Delta_{\Omega_2,\cdot}), \tag{18}$$

where $\Delta_{\Omega_1,\Omega_2}$ is the relative modular operator of the relative Tomita operator S_{Ω_1,Ω_2}, densely defined by

$$S_{\Omega_1,\Omega_2}\pi_2(A)\Omega_2 := \pi_1(A^*)\Omega_1 \quad \text{for} \quad A \in \mathcal{R}. \tag{19}$$

With (17), the operators $U(t)$ are the intertwiners between the two modular groups. They yield the called Radon-Nikodym cocycles $(D\omega_1 : D\omega_2)$, whence they are also denoted as $(D\omega_1 : D\omega_2)(t) := U(t)$. The cocycles satisfy the chain rule $(D\omega_1 : D\omega_2)(D\omega_2 : D\omega_3) = (D\omega_1 : D\omega_3)$.

If $\alpha_t^{\omega_1}$ is inner, it can be implemented by unitarities $e^{-itH_{\Omega_1}}$. Then (17) implies that there exists also a Hamiltonian H_{Ω_2}, such that the relative Hamiltonian (18) takes the form $H_{\Omega_1,\Omega_2} = H_{\Omega_1} - H_{\Omega_2}$, whence $(D\omega_1 : D\omega_2)$ is a coboundary in the group cohomology.

The outer modular automorphisms form the cohomology group $\text{Out}\mathcal{R} := \text{Aut}\mathcal{R}/\text{Inn}\mathcal{R}$ of modular automorphisms modulo inner modular automorphisms, characteristic for the types of factors contained in von Neumann algebra $\mathcal{R}$. Per definition $\text{Out}\mathcal{R}$ is trivial for inner automorphisms. Factors of type III_1 yield $\text{Out}\mathcal{R} = \mathbb{R}$.

Hence, in the case of thermal equilibrium states, corresponding to factors of type III_1 (see [14]), there is a distinguished 1-parameter group of outer modular automorphisms, which is a subgroup of the dynamical group.

Locking for a geometric interpretation for this subgroup, parametrized by $\mathbb{R}$, it should not be a coincidence that our partial order defined above could be parametrized by an open interval $]s_{\min}, s_{\max}[$, which is in fact diffeomorphic to $\mathbb{R}$. Therefore the dilations of the open sets should correspond to the 1-parameter group of outer modular automorphisms of thermal equilibrium states. The effect of large scale correlations thus becomes related to a thermal behavior of our localized states. A local equilibrium state might be characterized as a KMS state (see [12, 14]) over the algebra of observables on a double cone, whence (in agreement with the suggestions of [16]) the 1-parameter modular group in the KMS condition might be related to the time evolution. Note that, for double cones, a partial order may be induced from a split property of the algebras.

A geometric action of the modular group might be obtained by relating the thermal time to the geometric notion of dilations of the open sets. For any $x \in M$, the parameter s measures the extension of the sets $\mathcal{O}_s^x$. As accessability regions for a local measurement in M, these sets naturally increase with time. Hence it is natural to suggest that the parameter s might be related to the thermal time t. used to introduce a notion of time $t < s$ within a set $\mathcal{O}_s^x$.

For the ultralocal case (without UV cutoff), in [11] a construction of the causal structure for a space-time was bases on the corresponding net of operator algebras.

Nevertheless, let us for the moment still consider an apriori given underlying manifold M of the net. Locally around any point $x \in M$ we may induce open double cones as the pullback of the standard double cone, which in fact is the conformal model of Minkowski space. These open double cones then carry natural notions of time and causality, which are preserved under dilations. Therefore it seems natural to introduce locally around any $x \in M$ a causal structure and time by specializing the open sets to be open double cones $\mathcal{K}_s^x$ located at x, with timelike extension $2s$ between the ultimate past event p and the ultimate future event q involved in any measurement in $\mathcal{K}_s^x$ at x (time s between p and x, and likewise between x and q). Since the open double cones form a basis for the local topology of M, we might indeed consider equivalently the net of algebras located on open sets

$$\mathcal{O}_s^x := \mathcal{K}_s^x. \tag{20}$$

It is a difficult question, under which consistency conditions a local notion of time and causality might be extended, from nonzero environments of individual points to global regions. This will not be discussed here, but elsewhere [9].

However, if we assume the presence of factors of type III_1 in our von Neumann algebras, or likewise the existence of local equilibrium states, the choices for time and causality, made above on the basis of a partial order given by dilations which could algebraically be related to a commutant duality, are apparently natural.

References

[1] Ekstein H. 1967, Phys. Rev. 153, 1397-1402

[2] Rainer M. 1994, J. Math. Phys. 35, 64622

[3] Fredenhagen K. and Haag R. 1987, Commun. Math. Phys. 108, 91-115

[4] Salehi H. 1992, Class. Quant. Grav. 9, 2556

[5] Salehi H. 1996, Dynamical problems in generally covariant quantum field theory. IPM Preprint (in preparation)

[6] Haag R. and Kastler D. 1964, J. Math. Phys. 5, 848

[7] Bannier U. 1988, Commun. Math. Phys. 118, 163-170.

[8] Wollenberg M. 1990, On Causal Nets of Algebras, pp. 337-344 in: Operator Theory, Advances and Applications **43**, Linear Operators in Function Spaces, Proc. 12^{th} Int. Conf. on Operator Theory, Timişoara (Romania), June 6-16, 1988, Eds.: H. Helson, B. Sz.-Nagy, and F.-H. Vasilescu.

[9] Rainer M. and Salehi H. 1996, Causality and time on local nets of algebras with small and large scale cutoff limits. IPM Preprint (in preparation).

[10] Bannier U. 1994, Int. Jour. Theor. Phys. 33, 1797-1809.

[11] Keyl M. 1996, Rev. Math. Phys. 8, 229-270

[12] Bratteli O. and Robinson D. 1981, Operator algebras and quantum statistical mechanics I, Springer

[13] Baumgärtel H. and Wollenberg M. 1992, Causal nets of operator algebras, Berlin, Akademie Verlag.

[14] Haag R. 1992, Local quantum physics, Springer

[15] Connes A. 1995, Noncommutative Geometry, Academic Press.

[16] Connes A. and Rovelli C. 1994, Class. Quant. Grav. 11, 2899-2917.